# On the Marginal Instability of Linear Switched Systems

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***Abstract*— Stability properties for continuous-time linear switched systems are determined by the Lyapunov exponent associated with the system, which is the analogous of the joint spectral radius for the discrete-time case. This paper is concerned with the characterizations of stability properties when the Lyapunov exponent is zero. In this case it is well known that the system can be stable as well as unstable, even if it is never asymptotically stable nor it admits a trajectory blowing up exponentially fast. Our main result asserts that a switched system whose Lyapunov exponent is zero may be unstable only if a certain resonance condition is satisfied.**

## I. Introduction

We consider linear switched systems of the form

$$\dot{x}(t) = A(t)x(t)\,, \tag{1}$$

where $x \in \mathbf{R}^n$, $n \geq 2$, and the *switching law* $A(\cdot)$ is an arbitrary measurable function taking values on a compact and convex set of matrices $\mathcal{A} \subset \mathbf{R}^{n\times n}$. In the following, a switched system of the form (1) will be often identified with the corresponding set of matrices $\mathcal{A}$. This paper is concerned with stability issues for (1), where the stability properties are assumed to be uniform with respect to the switching law $A(\cdot)$.

A characterization of the stability behavior of $\mathcal{A}$ relies on the sign of the *(largest) Lyapunov exponent* associated with $\mathcal{A}$, which is defined as

$$\rho(\mathcal{A}) = \sup\left(\limsup_{t\to\infty}\frac{1}{t}\log\|x(t)\|\right), \tag{2}$$

where the $\sup$ is taken over the set of solutions of (1) with $\|x(0)\| = 1$ and $A(\cdot)$ is an arbitrary switching law. The Lyapunov exponent is a "measure" of the asymptotic stability of (1). Indeed the system is (uniformly) exponentially stable if and only if $\rho(\mathcal{A}) < 0$. That means that there exist $C_1, C_2 > 0$ such that, for every trajectory of (1) with $A(\cdot)$ an arbitrary switching law, one has

$$\|x(t)\| \leq C_1\exp(-C_2 t)\|x(0)\|, \quad t \geq 0.$$

On the other hand, (1) admits trajectories going to infinity exponentially fast if and only if $\rho(\mathcal{A}) > 0$. When $\rho(\mathcal{A}) = 0$, two situations may occur: $(i)$ all trajectories of (1) starting from a bounded set remain uniformly bounded and there exist trajectories staying away from the origin, in which case (1) is said to be *marginally stable*. $(ii)$ (1) admits a trajectory going to infinity and the system is said to be *marginally unstable*.

The role of the Lyapunov exponent is analogous to that of the joint spectral radius (or, equivalently, the generalized spectral radius) for discrete-time linear switched systems. The properties of the latter have been studied extensively in recent years (see for instance [1], [2], [3], [4]). In particular, for discrete-time linear switched systems, several results have been obtained in the case in which the spectral radius is equal to one under particular assumptions (see for instance [5] and references therein). This case corresponds to the situation $\rho(\mathcal{A}) = 0$ for continuous-time systems of the form (1).

The stability properties of continuous-time systems in the case $\rho(\mathcal{A}) = 0$ have not attracted much attention in the community up to now. Some results, relating marginal stability of (1) to the existence of limit cycles and periodic trajectories can be found in [6], [7], while some general observations about marginal stability and instability can be found in [8]. It has to be noted that a qualitative study of the properties of the trajectories in the case $\rho(\mathcal{A}) = 0$ leads to analogous properties for all values of $\rho$, since, as observed in [6], it holds $\rho(\mathcal{A}') = 0$, where $\mathcal{A}'$ is the set $\{A - \rho(\mathcal{A})\mathrm{Id} \,:\, A \in \mathcal{A}\}$ with $\mathrm{Id}$ denoting the $n \times n$ identity matrix.

The main result of the present paper, Theorem 2.7, states a necessary condition for marginal instability based on a *resonance* concept. Roughly speaking, if the switched system is marginally unstable then it must be reducible (see Definition 2.1 below), giving rise to a finite number of switched systems of lower dimensions, such that at least two of them are marginally stable and in resonance, i.e. they admit trajectories staying away from the origin associated with a common switching law (cf. Definition 2.6). Conversely, it turns out that the resonance phenomena highlighted in Theorem 2.7 do not guarantee marginal instability. Namely, the issue of understanding when marginal instability occurs appears in general to be very hard. We address this issue in the particular case in which $n \leq 4$ and $\mathcal{A}$ is the convex hull of $\{A^0, A^1\}$; we show that there are no non-trivial examples satisfying the resonance hypothesis for $n = 2, 3$ and that, for $n = 4$, for almost every choice of the matrices $A^0, A^1$ satisfying the resonance hypothesis, (1) admits a trajectory going to infinity with polynomial rate.

The rest of the paper is organized as follows. In Section II we provide the main notations and definitions of the paper, as well as the statement and the proof of our main result, Theorem 2.7. Section III establishes a sufficient condition for

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marginal instability in the particular case in which $n = 4$.

## II. Main result

The purpose of this section is to state and prove the main result of the paper. We first introduce some crucial definitions and preliminary results.

*Definition 2.1:* We say that

$$\{0\} = E_0 \subsetneq E_1 \subsetneq \cdots \subsetneq E_{k-1} \subsetneq E_k = \mathbf{R}^n \tag{3}$$

is an invariant flag for (1) if every $E_i$ is a subspace of $\mathbf{R}^n$ of dimension $n_i$, invariant with respect to every matrix $A \in \mathcal{A}$. An invariant flag is said to be *maximal* if, for every $i = 1, \ldots, k$, there does not exist a subspace V such that $E_{i-1} \subsetneq V \subsetneq E_i$ and $V$ is invariant with respect to $\mathcal{A}$. Finally an invariant flag is said to be *trivial* (resp. *nontrivial*) if $k = 1$ (resp. $k > 1$) and a switched system that admits (resp. does not admit) a nontrivial invariant flag is said to be *reducible* (resp. *irreducible*).

The following result relates the study of the stability properties of a reducible switched system to those of lower dimensional irreducible switched systems. The first part of the proposition is obvious while the second part can be easily checked by using suitable initial conditions and the variation of constant formula.

*Proposition 2.2:* Given a maximal invariant flag, there exists a vector basis $\{v_1, \ldots, v_n\}$ such that, for $i = 1, \ldots, k$, one has $E_i = \operatorname{span}\{v_1, \ldots, v_{n_i}\}$ and such that every matrix $A \in \mathcal{A}$ takes the following block form

$$A = \begin{pmatrix} A_{11} & A_{12} & \cdots & & \\ 0 & A_{22} & A_{23} & \cdots & \\ 0 & 0 & A_{33} & A_{34} & \cdots \\ \vdots & \ddots & \ddots & \ddots & \\ 0 & \cdots & \cdots & 0 & A_{kk} \end{pmatrix}, \tag{4}$$

where $A_{ij} \in \mathbf{R}^{(n_i - n_{i-1}) \times (n_j - n_{j-1})}$. In this case the *subsystems of* $\mathcal{A}$, defined as the switched systems corresponding to the sets $\mathcal{A}_i := \{A_{ii} : A \in \mathcal{A}\}$ for $i = 1, \ldots, k$, are irreducible and verify $\rho(\mathcal{A}_i) \le \rho(\mathcal{A})$ for $1 \le i \le k$, with equality holding for at least one index $i$.

Note that the choice of the invariant flag (3) uniquely determines the subsystems $\mathcal{A}_i$, up to changes of coordinates on $\mathbf{R}^n$ keeping the block form (4). Note, moreover, that with any switching law $A(\cdot)$ in $\mathcal{A}$ it is naturally associated a corresponding switching law $A_{ii}(\cdot)$ in $\mathcal{A}_i$, for $i = 1, \ldots, k$.

From [6] we have the following.

*Theorem 2.3:* If (1) is irreducible then there exists a norm $v(\cdot)$ in $\mathbf{R}^n$ such that, for every $x_0 \in \mathbf{R}^n$ and every trajectory $x(\cdot)$ of (1) starting from $x_0$, $v(x(t)) \le v(x_0)e^{\rho(\mathcal{A})t}$. Moreover, for every $x_0 \in \mathbf{R}^n$, there exists a trajectory of (1) starting from $x_0$ satisfying $v(x(t)) = v(x_0)e^{\rho(\mathcal{A})t}$.

The norm $v(\cdot)$ introduced in the previous theorem will be referred to as the *Barabanov norm*.

*Remark 2.4:* Notice that an immediate consequence of the previous result is the nontrivial observation that an irreducible switched system (1) with $\rho(\mathcal{A}) = 0$ must be stable. Indeed in that case we deduce that the balls with respect to the Barabanov norm are invariant for (1). On the other hand, for every initial condition $x_0$, there exists a trajectory of (1) lying on the sphere $v^{-1}(v(x_0))$.

*Remark 2.5:* Combining Proposition 2.2 with Theorem 2.3 and by a simple application of the variation of constant formula to the case where $\rho(\mathcal{A}) = 0$, we get that a trajectory can go to infinity at most polynomially. More precisely, there exists $C > 0$ such that, for every trajectory of (1) one has

$$\|x(t)\| \le C(1 + t^{k-1})\|x_0\|, \quad t \ge 0, \tag{5}$$

where $k$ is the number of subsystems associated with $\mathcal{A}$.

In the following, for $i = 1, \ldots, k$, we will denote by $v_i$ the Barabanov norm associated with the subsystem $\mathcal{A}_i$ and by $\mathcal{S}_i$ the corresponding unit sphere $v_i^{-1}(1)$.

*Definition 2.6:* Consider a reducible switched system $\mathcal{A}$ and denote by $\mathcal{A}_1, \ldots, \mathcal{A}_k$ the subsystems corresponding to a maximal invariant flag, as in Proposition 2.2. We say that two subsystems $\mathcal{A}_{i_1}, \mathcal{A}_{i_2}$, $i_1 \ne i_2$, of $\mathcal{A}$ are *in resonance* if they satisfy the following conditions

- $(a)$ $\rho(\mathcal{A}_{i_1}) = \rho(\mathcal{A}_{i_2}) = 0$ (thus $\mathcal{A}_{i_1}, \mathcal{A}_{i_2}$ are stable);
- $(b)$ there exists a switching law $A(\cdot)$ in $\mathcal{A}$ with associated switching laws $A_{i_j i_j}(\cdot)$ in $\mathcal{A}_{i_j}$ and two corresponding trajectories $\gamma_{i_j}(\cdot)$ of $\mathcal{A}_{i_j}$ such that $\gamma_{i_j}(t) \in \mathcal{S}_{i_j}$ for every $t > 0$ and for j=1,2.

We can now state the main result of the paper.

*Theorem 2.7:* Let $\mathcal{A}$ be a convex compact set of $n \times n$ matrices, $n \ge 2$. Assume that the linear switched system associated with $\mathcal{A}$ is marginally unstable. Then $\mathcal{A}$ is reducible and, for any maximal invariant flag, it admits two subsystems $\mathcal{A}_{i_j}$, $j = 1, 2$, in resonance.

*Proof:* We will prove the theorem by contradiction, i.e. by showing that, if there are no subsystems of $\mathcal{A}$ in resonance, then the system is stable. Assume that there exists a maximal invariant flag such that all the matrices of $\mathcal{A}$ have the form (4) and let $x = (x_1, \ldots, x_k)$ where $x_i \in \mathbf{R}^{n_i - n_{i-1}}$ for $i = 1, \ldots, k$. Consider a switching law $A(\cdot) \in \mathcal{A}$ and let $R_i(t, \tau)$, for $\tau, t \in \mathbf{R}$, be the resolvent of the time-varying linear system $\dot{z}_i = A_{ii} z_i$, $z_i \in \mathbf{R}^{n_i - n_{i-1}}$, i.e. $z_i(t) = R_i(t, \tau) z_i(\tau)$.

In particular we have $x_k(t) = R_k(t, 0) x_k(0)$ and, since

$$\begin{aligned} \dot{x}_{k-1}(t) &= A_{k-1,k-1}(t)\, x_{k-1}(t) + A_{k-1,k}(t)\, x_k(t) \\ &= A_{k-1,k-1}(t)\, x_{k-1}(t) + A_{k-1,k}(t)\, R_k(t,0)\, x_k(0)\,, \end{aligned}$$

by the variation of constant formula, we get

$$\begin{aligned} x_{k-1}(t) \ &= \ R_{k-1}(t,0) x_{k-1}(0) \\ &\quad + \int_0^t R_{k-1}(t,\tau) A_{k-1,k}(\tau)\, R_k(\tau, 0)\, x_k(0)\,. \end{aligned}$$

Repeating recursively the previous computations, we get

$$\begin{aligned} x_i(t) \ &= \ R_i(t,0) x_i(0) \\ &\quad + \sum_{h=1}^{k-i-1} \sum_{i < i_1 < \cdots < i_h \le k} I(t, i, i_1, \ldots, i_h) x_{i_h}(0), \end{aligned}$$

where the integral $I(t,i,i_1,\ldots,i_h)$ is defined as

$$\int_{t\geq\tau_1\geq\cdots\geq\tau_h\geq 0} R_i(t,\tau_1)A_{i,i_1}(\tau_1)\,R_{i_1}(\tau_1,\tau_2)\cdots \cdots A_{i_{h-1},i_h}(\tau_h)R_{i_h}(\tau_h,0)\,d\tau_1\cdots d\tau_h. \tag{6}$$

We will prove the proposition by estimating each integral $I(t,i,i_1,\ldots,i_h)$. We first introduce the following matrix norms. For $1\leq i\leq k$,

$$\|M\|_i := \max_{\substack{z\in\mathbf{R}^{n_i-n_{i-1}}\\ v_i(z)=1}} v_i(Mz)\,,\quad M\in\mathbf{R}^{(n_i-n_{i-1})\times(n_i-n_{i-1})},$$

where we recall that $v_i$ is the Barabanov norm associated with $\mathcal{A}_i$. Since two norms defined on finite dimensional vector spaces are always equivalent there exists $K_i>0$ such that $\|M\|\leq K_i\|M\|_i$ for $i=1,\ldots,k$, where $\|\cdot\|$ denotes the usual matrix norm. Moreover the norms $\|\cdot\|_i$ are sub-multiplicative norms, i.e., for every pair of matrices $M_1,M_2\in\mathbf{R}^{(n_i-n_{i-1})\times(n_i-n_{i-1})}$, one has $\|M_1\,M_2\|_i\leq\|M_1\|_i\,\|M_2\|_i$. Finally, by definition, they satisfy $\|R_i(\tau_1,\tau_2)\|_i\leq e^{\rho(\mathcal{A}_i)(\tau_1-\tau_2)}$ for every choice of the switching law and $0\leq\tau_2<\tau_1$. Since $\mathcal{A}$ is compact, we get

$$\begin{aligned}
&\|\textstyle\int_{t\geq\tau_1\geq\cdots\geq\tau_h\geq 0} R_i(t,\tau_1)A_{i,i_1}(\tau_1)\,R_{i_1}(\tau_1,\tau_2)\cdots\\
&\qquad\cdots A_{i_{h-1},i_h}(\tau_h)R_{i_h}(\tau_h,0)\,d\tau_1\cdots d\tau_h\|\\
&\leq \textstyle\int_{t\geq\tau_1\geq\cdots\geq\tau_h\geq 0}\|R_i(t,\tau_1)A_{i,i_1}(\tau_1)\,R_{i_1}(\tau_1,\tau_2)\cdots\\
&\qquad\cdots A_{i_{h-1},i_h}(\tau_h)R_{i_h}(\tau_h,0)\|\,d\tau_1\cdots d\tau_h\\
&\leq K\textstyle\int_{t\geq\tau_1\geq\cdots\geq\tau_h\geq 0}\|R_i(t,\tau_1)\|_i\,\|R_{i_1}(\tau_1,\tau_2)\|_{i_1}\cdots\\
&\qquad\cdots\|R_{i_h}(\tau_h,0)\|_{i_h}\,d\tau_1\cdots d\tau_h,
\end{aligned}$$

for a suitable $K>0$ independent of the switching law. Let us fix $T>0$ and assume, without loss of generality, that $t=mT$ for some positive integer $m$. Then, if we indicate the integer part of a real number with the symbol $[\cdot]$, we get

$$\begin{aligned}
&\int_{mT\geq\tau_1\geq\cdots\geq\tau_h\geq 0}\|R_i(mT,\tau_1)\|_i\,\|R_{i_1}(\tau_1,\tau_2)\|_{i_1}\cdots\\
&\qquad\cdots\|R_{i_h}(\tau_h,0)\|_{i_h}\,d\tau_1\cdots d\tau_h\\
&\leq K'\int_{mT\geq\tau_1\geq\cdots\geq\tau_h\geq 0}\|R_i(mT,\left[\frac{\tau_1}{T}\right]T)\|_i\,\|R_{i_1}(\left[\frac{\tau_1}{T}\right]T,\left[\frac{\tau_2}{T}\right]T)\|_{i_1}\\
&\qquad\cdots\|R_{i_h}(\left[\frac{\tau_h}{T}\right]T,0)\|_{i_h}\,d\tau_1\cdots d\tau_h\\
&\leq K'\sum_{0\leq m_h\leq\cdots\leq m_0=m}\|R_i(m_0T,m_1T)\|_i\,\|R_{i_1}(m_1T,m_2T)\|_{i_1}\\
&\qquad\cdots\|R_{i_h}(m_hT,0)\|_{i_h}\\
&\leq K'\sum_{0\leq m_h\leq\cdots\leq m_0=m}\prod_{j=1}^{m_h}\|R_{i_h}(jT,(j-1)T)\|_{i_h}\cdots\\
&\cdots\prod_{j=m_2+1}^{m_1}\|R_{i_1}(jT,(j-1)T)\|_{i_1}\prod_{j=m_1+1}^{m}\|R_i(jT,(j-1)T)\|_i,
\end{aligned}\tag{7}$$

for a suitable $K'\geq 1$ independent of the switching law. We want to prove that the previous sum is uniformly bounded with respect to the choice of the switching law and independently of $m$, at least when $T$ is large enough. To this purpose, we will need two preliminary results.

*Lemma 2.8:* Assume that there are no subsystems of $\mathcal{A}$ in resonance. Then, for $T$ large enough, there exists $C\in(0,1)$ such that, for every pair of distinct indices $(i,j)$ with $1\leq i,j\leq k$ and for every switching law,

$$\|R_i(T,0)\|_i\|R_j(T,0)\|_j\leq C\,.$$

*Proof:* If $\rho(\mathcal{A}_i)<0$ or $\rho(\mathcal{A}_j)<0$ then the thesis is true for every $T>0$. Therefore let us suppose without loss of generality that $i=1$, $j=2$ and $\rho(\mathcal{A}_1)=\rho(\mathcal{A}_2)=0$. Proceeding by contradiction, let us assume that there exist sequences of switching laws $A^{(n)}(\cdot)$, initial data $x_l^{(n)}(0)$ with $v_l(x_l^{(n)}(0))=1$ for $l=1,2$ and times $T^{(n)}$, with $\lim_{n\to\infty}T^{(n)}=\infty$ such that $v_l(x_l^{(n)}(T^{(n)}))>1-\frac{1}{n}$ for $l=1,2$, where $x_l^{(n)}(\cdot)$ is the solution of the switched system $\mathcal{A}_l$ corresponding to $A^{(n)}(\cdot)$. Since $\mathcal{A}$ is compact and convex, a classical result generalizing Banach-Alaoglu theorem establishes the existence of a weak-* limit of $A^{(n)}(\cdot)$ in $L^\infty([0,+\infty),\mathcal{A})$ (see for instance [9]) i.e., up to a subsequence, $A^{(n)}(\cdot)\overset{w^*}{\rightharpoonup}A^*(\cdot)$ in $L^\infty([0,+\infty),\mathcal{A})$. Thus $x_l^{(n)}(\cdot)\xrightarrow{L^\infty_{\rm loc}([0,\infty))}x_l^*(\cdot)$ for $l=1,2$, where $x_l^*(\cdot)$ is the solution of the switched system $\mathcal{A}_l$ corresponding to $A^*(\cdot)$ (see for instance [10]). In particular $v_l(x_l^*(t))=1$ for $t>0$ and $l=1,2$, contradicting the hypothesis of the lemma. ■

To prove that the sum (7) is uniformly bounded, we will use the following lemma.

*Lemma 2.9:* Let $h\in\mathbf{N}$, $h>1$. Let us define

$$\Xi_m=\{k\in\mathbf{N}^h\,:\,k_l\leq k_{l+1}\leq m\text{ for }l=1,\ldots,h-1\}.$$

Moreover, given a set of real numbers

$$\alpha=\{\alpha_i^l\in(0,1]\,:\,l=1,\ldots,h+1\,,\;i\in\mathbf{N}\}$$

and $k=(k_1,\ldots,k_h)\in\Xi_m$, let us define

$$\alpha_k=(\prod_{i=1}^{k_1}\alpha_i^1)(\prod_{i=k_1+1}^{k_2}\alpha_i^2)\cdots(\prod_{i=k_h+1}^{m}\alpha_i^{h+1})\,,$$

and

$$S_m(\alpha)=\sum_{k\in\Xi_m}\alpha_k\,.$$

Then, for any fixed $C\in(0,1)$, there exists a constant $L$ depending on $C$ such that, for every $m\in\mathbf{N}$ and for every set $\alpha$ of the previous form satisfying $\alpha_i^j\alpha_i^l\leq C\quad\forall j\neq l\,,\ \forall i\in\mathbf{N}$, one has $S_m(\alpha)\leq L$.

*Proof:* Let $k^{(1)},k^{(2)}\in\Xi_m$ and let us observe that

$$\alpha_{k^{(1)}}\alpha_{k^{(2)}}\leq C^{\max_{l=1,\ldots,h}|k_l^{(1)}-k_l^{(2)}|}. \tag{8}$$

Indeed, assume without loss of generality that $k_{l_*}^{(1)}<k_{l_*}^{(2)}$, where

$$|k_{l_*}^{(1)}-k_{l_*}^{(2)}|=\max_{l=1,\ldots,h}|k_l^{(1)}-k_l^{(2)}|.$$

Let $i$ be an integer verifying $k_{l_*}^{(1)}<i\leq k_{l_*}^{(2)}$. If $\alpha_i^{j_1}$ and $\alpha_i^{j_2}$ are terms corresponding to the subscript $i$ in the factorization of $\alpha_{k^{(1)}}$ and $\alpha_{k^{(2)}}$, respectively, it is then easy to see that

$j_2 \le l_* < j_1$, implying that $\alpha_i^{j_1}\alpha_i^{j_2} \le C$ from the hypothesis of the lemma. Thus (8) follows.

For $q \in \mathbf{N}$, let us define the set

$$\mathcal{I}_q = \{k \in \Xi_m \,:\, \alpha_k > C^q\},$$

and let us observe that, if $k^{(1)}, k^{(2)} \in \mathcal{I}_q$, then $\alpha_{k^{(1)}}\alpha_{k^{(2)}} > C^{2q}$ so that, from (8), we deduce that

$$\max_{l=1,\dots,h} |k_l^{(1)} - k_l^{(2)}| < 2q \qquad \forall k^{(1)}, k^{(2)} \in \mathcal{I}_q\,.$$

In particular this implies that the set $\mathcal{I}_q$ contains at most $(2q)^h$ elements. Since $\Xi_m = \cup_{q=1}^{\infty}(\mathcal{I}_q \setminus \mathcal{I}_{q-1})$, we get

$$S_m(\alpha) = \sum_{q=1}^{\infty} \sum_{\hat{k}\in\mathcal{I}_q\setminus\mathcal{I}_{q-1}} \alpha_{\hat{k}} \le \sum_{q=1}^{\infty}(2q)^h C^{q-1} < +\infty\,.$$

The lemma is proved by setting $L = \sum_{q=1}^{\infty}(2q)^h C^{q-1}$. ■

The proof of the theorem is then concluded in view of Lemma 2.8 and by applying Lemma 2.9 to the sum (7). ■

## III. SUFFICIENT CONDITIONS FOR MARGINAL INSTABILITY

A natural question arising from the results of the previous section is whether, or under which additional conditions, a switched system with $\rho(\mathcal{A}) = 0$ and which admits subsystems in resonance is marginally unstable. A simple observation is that if $\rho(\mathcal{A}) = 0$ and if we assume that there exists a vector basis such that each matrix of $\mathcal{A}$ can be put in the block form (4) with $A_{ij} = 0$ for $i < j$ then the switched system (1) is stable, independently of the existence or non-existence of subsystems in resonance. Indeed in this case, setting $x = (x_1, \dots, x_k)$ where $x_i \in \mathbf{R}^{n_i - n_{i+1}}$ for $i = 1, \dots, k$, the components $x_i$ of a trajectory of (1) vary independently and the stability of the overall system is therefore guaranteed by the fact that $\rho(\mathcal{A}_i) \le 0$. The role of the interaction terms $A_{ij}$ is therefore fundamental to possibly show the existence of trajectories going to infinity. In the general case, a complete analysis of the contribution of these interaction terms is definitely a hard issue to address. Therefore we will limit ourselves to the rather explicit case where $n = 4$ and $\mathcal{A}$ is the convex hull of two matrices $A^0$ and $A^1$, denoted by $\mathrm{co}\{A^0, A^1\}$.

*Definition 3.1:* A marginally unstable switched system associated with a convex compact subset $\mathcal{A}$ of $n \times n$ matrices is said to be *polynomially unstable of degree* $l$ if there exists a positive integer $l$ and constants $C_1, C_2 > 0$ such that every solution $x(\cdot)$ of (1) verifies $\|x(t)\| \le C_1(1+t^l)\|x(0)\|$ and there exists a solution $\bar{x}(\cdot)$ of (1) satisfying $\|\bar{x}(t)\| \ge C_2 t^l \|\bar{x}(0)\|$ for every $t > 0$.

### *A. The four dimensional linear switched systems*

Let $\mathcal{A} = \mathrm{co}\{A^0, A^1\}$, where $A^0, A^1$ are $n \times n$ matrices be a marginally unstable system. We consider the particular case in which all the matrices $uA^0 + (1-u)A^1$, where $u \in [0,1]$, are Hurwitz.

Thus, as a consequence of Theorem 2.7, it turns out immediately that $n \ge 4$, since otherwise one of the two subsystems obtained by applying Theorem 2.7 would be of

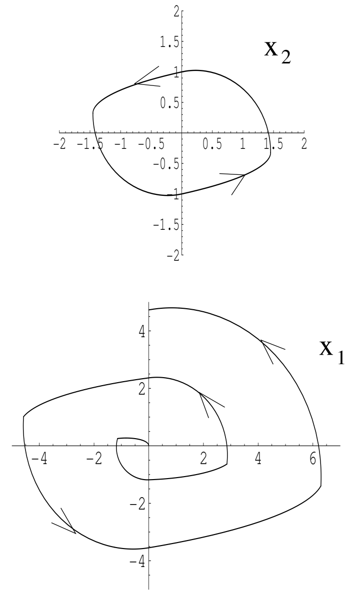


Fig. 1. Example 3.2: Trajectory of a polynomially unstable switched system

dimension one and its maximal Lyapunov exponent would be equal to zero, leading to one of the two matrices $A^0, A^1$ having 0 as eigenvalue.

Let us fix $n = 4$. Marginal instability may appear only when there is a maximal invariant flag (3) with $k = 2, n_1 = 2$ and, for the associated subsystems, $\rho(\mathcal{A}_1) = \rho(\mathcal{A}_2) = 0$. In particular we can write the matrices $A^0, A^1$ in the block form

$$A^0 = \begin{pmatrix} A_{11}^0 & A_{12}^0 \\ 0 & A_{22}^0 \end{pmatrix}, \quad A^1 = \begin{pmatrix} A_{11}^1 & A_{12}^1 \\ 0 & A_{22}^1 \end{pmatrix}, \qquad (9)$$

where all the blocks are $2 \times 2$. We know from [11], [12] that the planar switched systems $\mathcal{A}_* = \mathrm{co}\{A_*^0, A_*^1\}$, with $\rho(\mathcal{A}_*) = 0$ and such that all the matrices of $\mathcal{A}_*$ are Hurwitz, are those admitting a closed worst trajectory. A closed worst trajectory corresponds to a periodic switching law with $A_*(t) = A_*^0$ on time intervals of length $t_0 > 0$ and $A_*(t) = A_*^1$ on time intervals of length $t_1 > 0$. In the following $t_0, t_1$ will be called *switching times*. It turns out that the period of the closed worst trajectory is equal to $2t_0 + 2t_1$, i.e. the worst trajectory is the concatenation of four *bang arcs*. For simplicity let us denote by $T$ the semiperiod $t_0 + t_1$.

It is now easy to build an example of polynomially unstable switched system with matrices in the block form (9).

*Example 3.2:* Assume that $A^0, A^1$ in (9) are such that $A_{11}^0 = A_{22}^0 = A_*^0$, $A_{11}^1 = A_{22}^1 = A_*^1$ and $A_{12}^0 = A_{12}^1 = \mathrm{Id}$, where $\mathcal{A}_* = \mathrm{co}\{A_*^0, A_*^1\}$ satisfies the previous properties. If the resolvent $R_*(t,0)$ corresponds to a worst switching strategy for $\mathcal{A}_*$, as defined above, one can immediately verify, with the variation of constant formula, that

$$\begin{aligned} x_2(t) &= R_*(t,0)x_2(0)\,, \\ x_1(t) &= R_*(t,0)x_1(0) + t\,R_*(t,0)x_2(0)\,, \end{aligned}$$

so that the system is polynomially unstable of degree 1. An explicit numerical example can be obtained by considering the matrices

$$A^0_* = \begin{pmatrix} -1 & -\alpha \\ \alpha & -1 \end{pmatrix}, \qquad A^1_* = \begin{pmatrix} -1 & -\alpha \\ 1/\alpha & -1 \end{pmatrix}.$$

For a value $\alpha \sim 4.5047$ one has $\rho(\mathcal{A}_*) = 0$. Figure 1 depicts a particular trajectory for such a value.

*Remark 3.3:* The combined results of Proposition 2.2 and Theorem 2.7 are in the same spirit as Lemma 1 in [8], where the author states that a marginally unstable system admits a (possibly non-maximal) proper invariant flag of length two identifying two subsystems $\mathcal{A}_1$, $\mathcal{A}_2$ with $\rho(\mathcal{A}_1) = \rho(\mathcal{A}_2) = 0$. It should be noticed that the conclusion in [8, Lemma 1] goes a bit further by stating that both $\mathcal{A}_1$ and $\mathcal{A}_2$ can be taken marginally stable. However the latter statement is not true in general. Indeed, let

$$A^1 = A^2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then the only proper invariant subspaces of the system $\mathrm{co}\{A^1, A^2\}$ are $\mathbf{R} \times \{(0,0)\}$ and $\mathbf{R}^2 \times \{(0)\}$. It is easy to check that in this case for both possible invariant flags and every possible choice of compatible coordinates one of the subsystems is marginally unstable.

Assume that $t \mapsto R_*(t,0)\,\bar{x}_1$ is a closed worst trajectory where the resolvent $R_*(t,0)$ corresponds to the switching law

$$A_*(t) = \begin{cases} A^0_* & \text{if } t \in \big[kT, kT + t_1\big),\ k \in \mathbf{N}, \\ A^1_* & \text{if } t \in \big[kT + t_1, (k+1)T\big),\ k \in \mathbf{N}. \end{cases} \tag{10}$$

It turns out that $\bar{x}_1$ is an eigenvector of the matrix $R_*(T,0)$ corresponding to the eigenvalue $-1$. Let us denote by $\bar{x}_2$ an eigenvector corresponding to the second eigenvalue of $R_*(T,0)$ and, for $x \in \mathbf{R}^2$, by $\Pi_{\bar{x}_1}(x)$ the component of the vector $x$ along the direction $\bar{x}_1$ with respect to the basis $\{\bar{x}_1, \bar{x}_2\}$ of $\mathbf{R}^2$.

*Theorem 3.4:* Let $A^0, A^1$ be two $4 \times 4$ Hurwitz matrices in the block form (9). We use $A_{12}(\cdot)$ to denote the top right $2 \times 2$ block in the switching law $A(\cdot)$. Assume that

$(C1)$ the switched systems $\mathcal{A}_1 = \mathrm{co}\{A^0_{11}, A^1_{11}\}$ and $\mathcal{A}_2 = \mathrm{co}\{A^0_{22}, A^1_{22}\}$ admit closed worst trajectories with switching times $t^1_0, t^1_1$ and $t^2_0, t^2_1$, respectively,

$(C2)$ $t^1_0 = t^2_0 =: t_0$, $t^1_1 = t^2_1 =: t_1$,

$(C3)$ the condition

$$\int_0^T \Pi_{x^1_1}\Big(R_1(T,\tau) A_{12}(\tau) R_2(\tau,0)\, x^2_1\Big)\, d\tau \neq 0$$

is satisfied. Here $R_k(\cdot,\cdot)$, $k = 1,2$, are the resolvents associated with the time-varying systems defining the worst trajectories, as in (10), and, similarly to what done before, $T = t_0 + t_1$, $x^k_1, x^k_2$ are eigenvectors of $R_k(T,0)$, for $k = 1,2$, and $\Pi_{x^1_1}(x)$ is the component of the vector $x$ along the direction $x^1_1$ with respect to the basis $\{x^1_1, x^1_2\}$.

Then (1) is polynomially unstable of degree one. Moreover assume that $\mathcal{A}_1$ and $\mathcal{A}_2$ verify conditions $(C1)$, $(C2)$. Then, there exists a subset of pairs of matrices $(A^0_{12}, A^1_{12})$ which is open and dense in $M_2(\mathbf{R}) \times M_2(\mathbf{R})$, such that condition $(C3)$ is verified.

*Proof:* Let us consider the trajectory of (1) starting at $(0, x^2_1)^T$ and corresponding to the worst switching strategy for $\mathcal{A}_1, \mathcal{A}_2$,

$$x_2(t) = R_2(t,0) x^2_1 \tag{11}$$

$$x_1(t) = \int_0^t R_1(t,\tau) A_{12}(\tau) R_2(\tau,0) x^2_1 d\tau. \tag{12}$$

We first prove that the system is polynomially unstable under the hypotheses of the theorem. Fix $\tau \in [0,t]$ and consider the quotient $q(\tau) = \left[\frac{\tau}{T}\right]$ and the remainder $r(\tau) = \tau - T\left[\frac{\tau}{T}\right]$. Notice that

(I) the matrix $A_{12}(\cdot)$ only depends on $r(\tau)$, i.e.
$$A_{12}(\tau) = A_{12}(r(\tau)),$$

(II) $R_1(\tau_1, \tau_2) = R_1(\tau_1 + T, \tau_2 + T)$ and $R_2(\tau_1, \tau_2) = R_2(\tau_1 + T, \tau_2 + T)$ for every $\tau_1, \tau_2 > 0$, since the period of the switching law is $T$,

(III) for all $0 \le \tau \le mT,\ m \in \mathbf{N}$, we can write
$$R_1(mT,\tau) = R_1\big(mT, (q(\tau)+1)T\big)\, R_1\big((q(\tau)+1)T, \tau\big),$$
$$R_2(\tau,0) = R_2\big(\tau, q(\tau)T\big)\, R_2\big(T,0\big)^{q(\tau)},$$

(IV) by definition of $\Pi_{x^1_1}$, if $v = v_1 x^1_1 + v_2 x^1_2 \in \mathbf{R}^2$, we have
$$\begin{aligned}\Pi_{x^1_1}\big(R_1(T,0)\,v\big) =& v_1\, \Pi_{x^1_1}\big(R_1(T,0)\,x^1_1\big) \\ &+ v_2\, \Pi_{x^1_1}\big(R_1(T,0)\,x^1_2\big) \\ =& -v_1 = -\Pi_{x^1_1}(v).\end{aligned}$$

Combining these facts we easily get

$$\begin{aligned}&\Pi_{x^1_1}\Big(\int_0^{mT} R_1(mT,\tau)\, A_{12}(\tau)\, R_2(\tau,0)\, x^2_1\, d\tau\Big) \\ &\quad = (-1)^{m-1} m \int_0^T \Pi_{x^1_1}\Big(R_1(T,\tau)\, A_{12}(\tau)\, R_2(\tau,0)\, x^2_1\Big)\, d\tau.\end{aligned}$$

Then, under the hypothesis $(C3)$, $|\Pi_{x^0_1}(x_1(mT))| \ge C_1 m$, so that $\|x(mT)\| \ge C_2 m$ and, by the continuity of the resolvent of (1), we easily get $\|x(t)\| \ge C_3 t$, for suitable strictly positive constants $C_1, C_2, C_3$. The proof of the first part of the theorem is complete. We are left to prove that condition $(C3)$ is is verified in a open and dense subset of $M^2(\mathbf{R}) \times M^2(\mathbf{R})$ (in particular it is verified generically with respect to the choice of the matrices $(A^0_{12}, A^1_{12})$).

*Lemma 3.5:* With the notations above, the linear map

$$\begin{aligned}\Psi : M^2(\mathbf{R}) \times M^2(\mathbf{R}) &\to M^2(\mathbf{R}) \\ (A^0_{12}, A^1_{12}) &\mapsto \int_0^T R_1(T,\tau)\, A_{12}(\tau)\, R_2(\tau,0)\, d\tau\end{aligned}$$

is onto.

*Proof:* Since the resolvents are obtained by composing exponential matrices we easily get

$$\begin{aligned}\Psi(A^0_{12}, A^1_{12}) =& e^{t_1 A^1_{11}} e^{t_0 A^0_{11}} \Big(\int_0^{t_0} e^{-\tau A^0_{11}} A^0_{12} e^{\tau A^0_{22}} d\tau\Big) \\ &+ e^{t_1 A^1_{11}} \Big(\int_0^{t_1} e^{-\tau A^1_{11}} A^1_{12} e^{\tau A^1_{22}} d\tau\Big) e^{t_0 A^0_{22}},\end{aligned}$$

and, by considering the special case $A^1_{12} = 0$, we are reduced to study the surjectivity of the linear map

$$A^0_{12} \mapsto \Psi_1(A^0_{12}) := \int_0^{t_0} e^{-\tau A^0_{11}} A^0_{12} e^{\tau A^0_{22}} d\tau \,.$$

Assume that $A^0_{12} \in \ker \Psi_1$. Then in particular

$$\begin{aligned} 0 &= -A^0_{11}\Psi_1(A^0_{12}) + \Psi_1(A^0_{12})A^0_{22} \\ &= \int_0^{t_0} \frac{d}{d\tau}\left(e^{-\tau A^0_{11}} A^0_{12} e^{\tau A^0_{22}}\right) d\tau \\ &= e^{-t_0 A^0_{11}} A^0_{12} e^{t_0 A^0_{22}} - A^0_{12}\,. \end{aligned}$$

Therefore, if we set $\tilde\Psi(A^0_{12}) = A^0_{12} e^{t_0 A^0_{22}} - e^{t_0 A^0_{11}} A^0_{12}$ we have that $\ker \Psi_1 \subseteq \ker \tilde\Psi$.

Let us denote by $\sigma(A^0_{11})$ and $\sigma(A^0_{22})$ the spectra of $A^0_{11}$ and $A^0_{22}$, respectively. Then the spectrum of $\tilde\Psi$ turns out to be

$$\sigma(\tilde\Psi) = \{e^{t_0\lambda} - e^{t_0\mu} \,:\, \lambda \in \sigma(A^0_{22}), \mu \in \sigma(A^0_{11})\}\,. \quad (13)$$

For reasons of space here we will just consider the case in which $\sigma(A^0_{11})$ and $\sigma(A^0_{22})$ are real. If $\sigma(A^0_{11}) \cap \sigma(A^0_{22}) = \emptyset$ we have $\ker\tilde\Psi = \ker\Psi_1 = \{0\}$. Assume now that $\sigma(A^0_{11}) \cap \sigma(A^0_{22}) = \{\lambda\}$. Let $\{v_1, w_1\}$ and $\{v_2, w_2\}$ vector bases such that $A^0_{11}$ and $A^0_{22}$ are in Jordan form ($v_1, v_2$ are eigenvectors associated with the eigenvalue $\lambda$). Notice that the cases $A^0_{11} = \lambda\,\mathrm{Id}$ and $A^0_{22} = \lambda\,\mathrm{Id}$ can be trivially excluded from our analysis. Consider a matrix $C \in \ker\tilde\Psi$. Then

$$\begin{aligned} 0 = \tilde\Psi(C)v_2 &= Ce^{t_0 A^0_{22}} v_2 - e^{t_0 A^0_{11}} C v_2 \\ &= e^{t_0\lambda} C v_2 - e^{t_0 A^0_{11}} C v_2 \end{aligned}$$

and therefore $Cv_2 = \alpha v_1$ for some $\alpha \in \mathbf{R}$. Also,

$$0 = \tilde\Psi(C)w_2 = Ce^{t_0 A^0_{22}} w_2 - e^{t_0 A^0_{11}} C w_2.$$

If $A^0_{22}$ is diagonalizable, the above expression is equal to $e^{t_0\mu} C w_2 - e^{t_0 A^0_{11}} C w_2$ with $\mu \notin \sigma(A^0_{11})$. It implies that $Cw_2 = 0$. When $A^0_{22}$ is non-diagonalizable, one has $0 = e^{t\lambda}(Cw_2 + \alpha t_0 v_1) - e^{t_0 A^0_{11}} C w_2$. Since $A^0_{11}$ is diagonalizable (in particular there exists $\mu \neq \lambda,\ \mu \in \sigma(A^0_{11})$), writing the previous expression in the basis $\{v_1, w_1\}$, we get

$$\begin{pmatrix} 0 & 0 \\ 0 & e^{t_0\lambda} - e^{t_0\mu} \end{pmatrix} Cw_2 + \alpha t_0 e^{t_0\lambda} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0\,,$$

which implies $\alpha = 0$ and $Cw_2 = \beta v_1$ for some $\beta \in \mathbf{R}$. If both $A^0_{22}, A^0_{11}$ are non-diagonalizable then we have

$$\begin{pmatrix} 0 & t_0 e^{t_0\lambda} \\ 0 & 0 \end{pmatrix} Cw_2 + \alpha t_0 e^{t_0\lambda} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0\,,$$

so that $Cw_2 = \beta v_1 + \alpha w_1$. Notice that in the latter case, unlike the previous ones, $\ker\tilde\Psi$ is a subspace of $\mathbf{R}^{2\times 2}$ of dimension two.

Summing up, we have that $C \in \ker\tilde\Psi$ if

- $Cv_2 = \alpha v_1\,,\ Cw_2 = 0\,,$ for some $\alpha \in \mathbf{R}$ if $A^0_{22}$ is diagonalizable,
- $Cv_2 = 0\,,\ Cw_2 = \beta v_1$ for some $\beta \in \mathbf{R}$ if $A^0_{11}$ is diagonalizable and $A^0_{22}$ is non-diagonalizable,
- $Cv_2 = \alpha v_1\,,\ Cw_2 = \beta v_1 + \alpha w_1$ for some $\alpha, \beta \in \mathbf{R}^2$ if $A^0_{11}, A^0_{22}$ are is non-diagonalizable.

Let us verify that, when $A^0_{11}, A^0_{22}$ are non-diagonalizable, any $C \in \ker\tilde\Psi$ does not belong to $\ker\Psi_1$. We have that

$$\begin{aligned} 0 &= \Psi_1(C)v_2 = \int_0^{t_0} e^{-\tau A^0_{11}} C e^{\tau A^0_{22}} v_2 d\tau \\ &= \alpha \int_0^{t_0} e^{-\tau A^0_{11}} e^{\lambda\tau} v_1 d\tau = \alpha t_0 v_1 \ \Rightarrow\ \alpha = 0\,, \end{aligned}$$

$$\begin{aligned} 0 &= \Psi_1(C)w_2 = \int_0^{t_0} e^{-\tau A^0_{11}} C e^{\tau A^0_{22}} w_2 d\tau \\ &= \int_0^{t_0} e^{-\tau A^0_{11}} A_{12}(e^{\lambda\tau} v_2 + \tau e^{\lambda\tau} w_2) d\tau \\ &= \beta \int_0^{t_0} \tau e^{-\tau A^0_{11}} e^{\lambda\tau} v_1 d\tau = \beta \frac{t_0^2}{2} v_1 \Rightarrow\ \beta = 0\,. \end{aligned}$$

Therefore $\ker\Psi_1 = \{0\}$. We skip the cases in which at least one among $A^0_{11}, A^0_{22}$ is diagonalizable, that can be treated similarly. ■

## IV. Conclusion

In this paper we showed a necessary condition for marginal instability of linear switched systems based on resonance properties of particular subsystems. In addition we proved that the switched systems defined by two linear dynamics in dimension four and satisfying this condition generically admit trajectories going to infinity at polynomial rate.

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